Journal of Intelligent Decision and Computational Modelling

www.jidcm.reapress.com

J. Intell. Decis. Comput. Model. Vol. 1, No. 2 (2025) 65–76.

Paper Type: Original Article

Portfolio Directional Distance Function Models with a Stochastic Jump Process

Mirsadeghpour Zoghi Seyedeh Masoumeh[1,*], Banihashemi Shokoofeh[2], Modarresi Navideh[2]

[1] Department of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran; maasoumeh.mirsadeghpour@gmail.com.

[2] Department of Mathematics, Allameh Tabataba'i University, Tehran, Iran; Shbanihashemi@atu.ac.ir; namomath@aut.ac.ir.

Citation:

Received: 14 November 2024 *Revised: 12 January 2025* *Accepted: 17 March 2025*	Mirsadeghpour Zoghi, S. M., Banihashemi, S., & Modarresi, N. (2025). Portfolio directional distance function models with a stochastic jump process. *Journal of intelligent decision and computational modelling, 1*(2), 65-76.

Abstract

Portfolio optimization problems include the selection of different assets to invest in order to maximize the return and minimize the risk. In practice, the models account for asset returns that skewness, Kurtosis, and heavy tails characterize. For this purpose, we describe the dynamics of assets' returns with the Variance Gamma (VG) process from Lévy processes by considering the skewness and Kurtosis of the assets' return rate. We employ Data Envelopment Analysis (DEA) methodology alongside Directional Distance Function (DDF), which they able to evaluate different assets' performance by VG process through its constraints, and they identify inefficiencies within asset markets. We introduce two models that seek to simultaneously minimize the risk measure as the input and maximize the mean return as the output of the given asset using the pre-specified direction vector. In the first model, stricter assessments arise from directions emphasizing maximum conditional risk and return. In the second model, mean return-risk values remain constant, suggesting that asset inefficiency is unaffected by changing directions. This unchanging pattern may reflect similar impacts across scenarios or limitations in the mean return-risk metric's ability to detect directional variations. Instead, asset inefficiency appears to be driven by intrinsic distributional properties, notably skewness and Kurtosis, rather than scenario-specific influences. An empirical example in the Iranian stock market of seven companies is used to validate the models.

Keywords: Data envelopment analysis, Portfolio optimization, Variance gamma process, Directional distance function.

1 | Introduction

In the financial world, investors and financial managers are always looking for ways to increase returns and minimize risks in their investment portfolios, so that portfolio evaluation can be necessary. One of the standard methods in this field is portfolio optimization, which determines the optimal allocation of a limited combination of assets for investment using various models to achieve high return and less uncertainty (risk).

Corresponding Author: maasoumeh.mirsadeghpour@gmail.com
https://doi.org/10.48314/jidcm.v1i2.65

Markowitz pioneered in the study of portfolio selection and mean-variance analysis [1], [2]. Data Envelopment Analysis (DEA) has been widely used as a non-parametric method for evaluating the efficiency of Decision-Making Units (DMUs) in various fields, including finance, healthcare, education, and manufacturing. Since its introduction [3], DEA has evolved into a powerful tool for assessing performance based on multiple inputs and outputs.

This non-parametric efficiency method has gained attention as a powerful tool for evaluating financial performance and investment efficiency [4]. This method allows for the comparison of different mutual funds and helps investors identify efficient assets. DEA has been applied in financial markets to evaluate the efficiency of investment portfolios. Researchers have explored how DEA can be used to preselect efficient assets, optimize portfolio allocation, and assess risk-adjusted returns. Studies have demonstrated that DEA-based portfolio selection can outperform traditional methods such as mean-variance optimization [5], [4]. Also, some approaches have presented a DEA-like structure in a mean-variance-skewness framework [6–8].

However, a significant challenge in using DEA is the lack of directionality in optimization in some classical models. To address this limitation, the Directional Distance Function (DDF) has been introduced as an advanced approach in DEA. This function enables investors to incorporate their preferences in portfolio optimization, allowing them to optimize based on specific directions rather than traditional models. DDF has been introduced for simultaneous input reduction and output expansion [9]. To estimate the value-based technical inefficiency, an approach has been proposed using the concept of DDF, which is generalized Shephard's distance function [10–12]. New cost, revenue, and profit-based measures of efficiency have been developed with respect to the DDF. These measures satisfy the property of translation invariance. Also, they can handle negative data by selecting the suitable direction vectors [13]. A non-radial DDF model that accommodates negative and flexible measures in DEA has been proposed, relaxing assumptions of non-negative data and fixed input-output roles. This study is crucial for extending DDF applications to scenarios with non-standard data structures [14]. A two-step methodology combining DEA with DDFs has been introduced to select efficient assets and interval multi-objective programming to optimize portfolio composition. This approach is valuable for its integration of DEA efficiency analysis with investor preferences in portfolio construction.

Risk in financial assets is a factor that influences the asset prices. Because of the tail and skewness in the distribution of asset returns, variance as a risk measure in Markowitz's theory is widely criticized by practitioners due to its symmetrical measure. Moreover, investors prefer a positive skewed distribution, i.e., a large chance of small loss [15]. Value-at-Risk (VaR) is another risk measure that is popular as an industry standard, but it is not always sub-additive nor convex. So, the concept of coherent risk measure is introduced that satisfies the properties of translation invariance, homogeneity, subadditivity, and monotonicity [16]. Conditional Value-at-Risk (CVaR) as an alternative and compatible risk measure is the weighted average of VaR and losses strictly greater than VaR for general distributions [17]. The method of CVaR minimization has been employed for credit risk management of a portfolio of bonds [18], and it has been applied in portfolio hedging [19], see also [20].

In the realm of dynamic portfolio selection, risky assets are commonly modeled using Brownian Motion with a normal distribution. However, empirical evidence often reveals that asset return distributions deviate from normality, exhibiting greater leptokurtosis and thicker tails than a normal distribution would suggest. To address these characteristics in portfolio optimization, the Variance Gamma (VG) process is employed, which accounts for leptokurtosis and skewness through its parameters [7], [21]. The VG process, characterized as a Brownian Motion with drift adjusted by a stochastic time change and Gamma-distributed variance, features independent and stationary increments starting from zero, making it suitable for modeling log asset prices and capturing models with infinite jumps within finite time intervals. Neglecting extreme events, such as asymmetric tail dependence, during portfolio construction may limit asset managers' ability to mitigate risk through diversification. Consequently, incorporating tail dependence into the covariance matrix has been proposed to enhance portfolio performance [22].

Furthermore, a hybrid accelerated simulation approach was developed for pricing Asian options with arithmetic average payoffs under the VG process, utilizing a decomposition of the payoff and importance sampling to reduce simulation variance [23]. Two optimization problems have been introduced, which they have integrated with the DEA based on the VG Lévy process, in which the input and output are stochastic. These models have assumed that the asset return follows a pure jump Lévy process [24].

In this paper, we aim to explore the connection between portfolio optimization and DEA and demonstrate how the DDF can enhance investment decision-making. Additionally, we will present mathematical models for evaluating asset efficiency performance under the VG process, helping investors make better decisions under price uncertainty. Risk is quantified as the only input with mean return as the output, to assess asset performance. VaR and CVaR are utilized to enhance risk measurement. This contribution has argued for formulating DEA models based on DDF that improve the efficiency of asset performance evaluation.

In financial modeling, minimizing risk while maximizing mean return constitutes a fundamental principle. Skewness and Kurtosis significantly influence risk (As an input) and mean return (As an output) of an asset, serving as critical metrics for assessing asset performance. These measures enhance the ability to evaluate the probability of extreme events in the tails of return distributions. To address these characteristics, the VG process is adopted, incorporating constraints on both inputs and outputs. The VG process offers several advantages, including its capacity to model heavy-tailed distributions, finite moments of all orders for reliable parameter estimation, and the ability to adjust skewness and Kurtosis through its parameters, thereby improving the accuracy of asset performance evaluations. In other words, the VG provides a better fit than usual for daily log-returns distributions. Therefore, we develop models employing a directional measure to simultaneously reduce risk and enhance mean return, assessing asset performance within a mean return-risk framework using the VG process. Three different direction vectors are considered to take the range of possible improvement in the input and/or output of the asset under evaluation [13]. By applying each pre-specified direction vector, the first model maximizes different proportional changes in the risk measure reduction and the augmentation in mean return. Any risk measure can utilize this model, and we use VaR and CVaR as the risk measures in the applied example. The second model in the mean return-CVaR framework uses former directions. In this model, the risk measure CVaR is applied in the form introduced in [25]. In both models, the efficiency of the asset under evaluation is characterized by its projection point and its distance from the efficient frontier. These models identify the extent to which risk must be reduced and returns increased to position an inefficient portfolio on the efficient frontier. In analyzing real data, the parameters of the VG process are first estimated using the method of moments. Subsequently, a Monte Carlo simulation is employed to generate VG process factors. The VG process is a better fit compared to the normal distribution. Additionally, by accounting for skewness and Kurtosis in asset performance evaluation, the VG process eliminates the need for supplementary constraints in the models. To validate the practicality of the proposed models, we apply seven companies of the Iran Stock Exchange Market from 2018 to 2019 [24].

The rest of the paper is organized as follows. In Section 2, we propose our VG-based models with DDFs in the mean return-risk framework. And the empirical example of the Iranian stock Exchange Market is provided in Section 3. In the conclusion section, we discuss our findings.

2 | Portfolio Directional Distance Function Models

In considerable empirical studies, asset returns distributions are non-normal, and they are more leptokurtic and exhibit skewness. In this section, we propose models based on a directional measure, which seek to simultaneously minimize the risk measure and maximize the mean return, evaluating an asset's performance in a mean return-risk framework under the VG process. As mentioned, skewness and Kurtosis are controlled by VG parameters in assessing. Therefore, this has an impact on the performance evaluation, directly, and it leads to more reliable efficiency scores. Our proposed models are based on the directional profit maximization problem [13]. It should be mentioned that the models proposed here are introduced in [24]. We suggest calling the models the Portfolio Directional Distance Function (PDDF). The input of the models is the risk measure, and the mean return is considered the output.

Let's suppose there are n financial assets. Return of each asset is defined as $Y^1, Y^1, ..., Y^n$. We consider $Y^o = (Risk^o, E(Y^o))^T$ as the asset under evaluation for $o \in 1,2,...,n$, $Risk^o$ is the risk measure, and $E(Y^o)$ is the mean return of Y^o. Let $g = (g_{Risk^o}, g_{E(Y^o)})^T$ be a direction vector, Risk as the risk measure and mean return, and for the asset under evaluation.

Now, we assume that the VG process describes the dynamics of assets' log returns. For this purpose, let $\{X_t^k, t \geq 0\}$ for $k = 1,2,...,n$ be the independent multiple VG factors $X_t^k \sim VG(t; \sigma_k, \theta_k, \nu_k)$, where σ_k, θ_k, ν_k are the parameters of the VG process. Also, we assume that S_t^j be the j-th asset's price process satisfies in

$$S_t^j = S_0^j \exp(\mu^j t + \sum_{k=1}^{n} a_{jk} X_t^k). \tag{1}$$

We build a portfolio with n financial assets $S^1, S^2, ..., S^n$ and the return of the j asset, i.e., $Y_t^j = LnS_t^j - LnS_0^j$ $(t \geq 0)$ follows:

$$Y_t^j = \mu^j t + \sum_{k=1}^{n} a_{jk} X_t^k. \tag{2}$$

Hereafter, for convenience, we use Y^j instead of Y_t^j. The following models measure the distance between the asset under evaluation and the efficient frontier. The input and the output of the model, the risk measure, and the mean return are estimated by VG parameters. We take three different direction vectors, consider taking the range of possible improvement in input and/or output as follows:

$$\begin{aligned} g_{Risk^o} &= Risk^o - \min_{j=1,2,...,n} \{Risk^j\}, \\ g_{E(Y^o)} &= \max_{j=1,2,...,n} \{E(Y^j)\} - E(Y^o). \end{aligned} \tag{3}$$

$$\begin{aligned} g_{Risk^o} &= \max_{j=1,2,...,n} \{Risk^j\}, \\ g_{E(Y^o)} &= \max_{j=1,2,...,n} \{E(Y^j)\}. \end{aligned} \tag{4}$$

$$\begin{aligned} g_{Risk^o} &= \max_{j=1,2,...,n} \{Risk^j\} - \min_{j=1,2,...,n} \{Risk^j\}, \\ g_{E(Y^o)} &= \max_{j=1,2,...,n} \{E(Y^j)\} - \min_{j=1,2,...,n} \{E(Y^j)\}. \end{aligned} \tag{5}$$

For each pre-specified direction vector g, we now present below the DDF-based DEA models in a portfolio framework under the VG process as the PDDF model.

$$\begin{aligned} \rho^*_{PDDF1} = \max \quad & g_{Risk^o} \alpha^- + g_{E(Y^o)} \alpha^+, \\ \text{s.t.} \quad & Risk(Y(\lambda)) \leq Risk^o - g_{Risk^o} \alpha^-, \\ & E(Y(\lambda)) \geq E(y^o) + g_{E(Y^o)} \alpha^+, \qquad Y^j = \mu^j + \sum_{k=1}^{n} a_{jk} X^k, \quad j = 1,2,...,n \\ & e^T \lambda = 1, \qquad \lambda \geq 0, \quad \alpha^- \geq 0, \alpha^+ \geq 0, \quad X^k \sim VG(t; \sigma_k, \nu_k, \theta_k). \quad k = 1,2,...,n. \end{aligned} \tag{6}$$

Model (6) is in the mean return-Risk framework that is the same as the BCC model in DEA, but it is under the VG process. The proportions of initial capital are shown by vector $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)^T$ in which the invested money in the asset j is λ_j. It is defined as a decision vector in *Model (6)* and e is the all-ones vector. The return of a portfolio is defined by $Y(\lambda)=\sum_{j=1}^{n}\lambda_j Y^j$ and the mean return of the portfolio is computed as

$$E(Y(\lambda))=\sum_{j=1}^{n}\lambda_j E(Y^j).$$

To solve the optimization problem, first of all, we compute Y^j for $j=1,2,\ldots,n$, through VG parameters and the Monte Carlo simulation technique, then we solve *Model (6)*. It should be noted that Y^j is the sample return of jth asset. First, we estimate the parameters of VG, the estimation method we employ is the Moment Estimation [21]. Then, the non-linear equations based on the first four central moments and covariance of returns distribution over the length of the given time t are obtained, they are solved by the Gauss-Newton algorithm, and the VG parameters are estimated. We apply the technique that is employed in [20]. Next, we simulate the VG factors $X^1,X^2,\ldots,X^n$ using the Monte Carlo method. Lastly $j=1,2,\ldots,n$, scenarios of financial assets log return are computed according to *Eq. (2)*. The advantage of the *Model (6)* is that skewness and Kurtosis of the return distributions are considered into efficiency performance through VG parameters.

Furthermore, the mean return and risk measure are affected by the VG process. The optimal value of ρ^*_{PDDF1} seeks simultaneously to reduce the risk measure and improve the mean return of the asset under evaluation in the direction of vector g. In other words, for a given asset, the optimal objective value of the model indicates different maximum proportional changes in Risk and mean return of the asset, and the purpose is to maximize ρ_{PDDF1} in direction $g=(g_{Risk^o},g_{E(Y^o)})^T$ for the risk measure and mean return, separately for the asset under evaluation.

Essentially, ρ^*_{PDDF1} in *Model (6)* is a measure of the distance between the under-evaluation asset and the efficient frontier. In the direction of vector $g=(g_{Risk^o},g_{E(Y^o)})^T$, the evaluated asset's projection coordinates are determined by the right-hand sides of the inequality constraints of *Model (6)* evaluated in the optimal solution (i.e., $(Risk^o-\alpha^- g_{Risk^o},E(Y^o)+\alpha^+ g_{E(Y^o)})^T$). In an empirical example, the CVaR and VaR are applied as risk measures in *Model (6)*. So, other risk measures can be used instead of Risk the directions and *Model (6)*.

Now, we introduce a model similar to *Model (6)*, but the first constraint differs from that in *Model (6)*. In this mode, we apply the CVaR as the risk measure, and it is approximated by discrete points based on the approach proposed by [25]. Therefore, the introduced *Model (7),* by considering the asset under evaluation, is as follows

$$
\begin{aligned}
\rho^*_{PDDF2}= \ \max \quad & g_{CVaR^o_\beta}\alpha^- + g_{E(Y^o)}\alpha^+,\\
\text{s.t.} \quad & \Gamma+\frac{1}{(1-\beta)Q}\sum_{q=1}^{Q}(-\lambda^T Y_q-\Gamma)^+ \le CVaR^o_\beta-\alpha^- R_{CVaR^o_\beta},\\
& E\big(Y(\lambda)\big)\ge E\big(Y^o\big)+\alpha^+ R_{E(Y^o)},\\
& Y^j=\mu^j+\sum_{k=1}^{n}a_{jk}X^k,\\
& j=1,2,\ldots,n,\\
& e^T\lambda=1, \quad \lambda\ge 0, \quad \alpha^-\ge 0,\ \alpha^+\ge 0,\\
& X^k\sim VG(t;\sigma_k,\nu_k,\theta_k), \quad k=1,2,\ldots,n.
\end{aligned} \qquad (7)
$$

As mentioned before, the model tries to measure the distance between the asset under evaluation and the projection point on the efficient frontier with different proportions. The risk measure is reduced by α^- while the mean return is increased by α^+. The optimal value of the model is $g_{CVaR^o_\beta}\alpha^- + g_{E(Y^o)}\alpha^+$ that it shows the inefficiency score of the asset under evaluation in the direction $g=(g_{CVaR_{\beta}^{o}},g_{E(Y^{o})})^{T}$. To solve the problem, first, the VG distribution factors $X^1,...,X^n$ are simulated by the Monte Carlo method. Then the scenarios of assets' log return, $Y_q=(Y_q^1,...,Y_q^n)^T$ for $q=1,...,Q$, are obtained by the simulated VG factors. The technique introduced in [23] is applied to solve the non-linear form of CVaR in the first constraint. So, the left hand of the first constraint is approximated by discrete points $Y_q=(Y_q^1,...,Y_q^n)^T$ for $q=1,...,Q$ which they are vectors in the space X^n, and they are simulated by VG factors, and it is approximated based on the Monte Carlo simulation technique. The mean return of the model is the second constraint in which $E(Y(\lambda))=\sum_{q=1}^{Q}\lambda^T E(Y_q)$ and it is the weighted average of mean returns. It is noted that CVaR is also calculated by the scenarios of the assets' log return, too.

The optimal objective value of the model indicates different maximum proportional changes in CVaR and mean return of the asset, and the purpose is to maximize ρ_{PDDF2} in directions of risk measure and mean return, separately. If the optimal value equals zero, then the asset under evaluation is just part of the efficient frontier [13]. Otherwise, each α^- and α^+ indicates the changes in CVaR and mean return of the asset under evaluation that guarantees the projected point of the asset is on the efficient frontier.

The risk measure VaR could not be applied in *Model (7)*, since then the model becomes the same as *Model (6)* in the mean return-VaR framework. Therefore, for the second model, we are not able to use VaR as the risk measure.

In both models, the asset under evaluation is called PDDF-efficient if $\rho_{PDDF}^{*}=0$. However, if $\rho_{PDDF}^{*}>0$, then the asset is PDDF-inefficient, and ρ_{PDDF}^{*} represents a change in risk measure and mean return that results in a projection of the evaluated asset onto the efficient frontier. Therefore, $1-\rho_{PDDF}^{*}$ shows the efficiency score of the asset under evaluation.

The models seek to simultaneously minimize the risk measure as the input and maximize the mean return as the output of the given asset using the pre-specified direction vector. The objective function can reflect the trade-offs between input and output. The PDDF-inefficient measures based on the *Direction Vectors (3)*, *(4),* and *(5)* are naturally non-radial (Non-proportional). The non-negativity restrictions on the α^- and α^+ have been imposed to make an asset become PDDF-efficient by reducing its risk, and by increasing its mean return. The models are VG-based, which they are consider skewness and Kurtosis in performance evaluation, so there is no need to consider any constraint for these characteristics separately in the models [24]. It is obvious how the optimal value of the models is dependent on the direction vector g.

3 | The Empirical Illustration

We consider the dataset of the Iran Stock Exchange as a case study. All of the share prices on the stock market are publicly available from the official website of Tehran Stock Exchange Market (TSE) [26]. To validate the practicality of the proposed models, seven companies are chosen for the 2018-2019 period [24]. The companies' names are Iran National Copper Industries (MSMI), Iran Khodro (IKCO), Mapna (MAPN), Arman Ati Mes ETF, Asan Pardakht Persian (APPE), Isfahan Steel Company (ZOBI), and Spahan Naft (SEPP).

Each company is considered a financial asset. The confidence level is considered $\beta=0.90$. We use GAMS and MATLAB software to do the computations. To compute the inefficiency score of each asset, first, let's show the skewness and Kurtosis of each asset in *Table 1*.

Table 1. Skewness and Kurtosis of each asset.

ASSET	SKEWNESS	KURTOSIS
MSMI	-2.0873	27.6953
IKCO	-0.0088	2.4636
MAPN	-8.8993	110.8949
SMIF	1.0106	9.4556
APPE	-4.6107	52.8753
ZOBI	-6.755	83.3473
SEPP	-0.2091	4.0696

As shown in *Table 1*, financial returns' high leptokurtosis and nonzero skewness are far from the Normal process, which confirms they have heavy tails. To overcome these restrictions, we adopt the VG process that controls skewness and Kurtosis, and it is also heavy-tailed. In *Fig. 1*, the stock returns of seven companies are displayed.

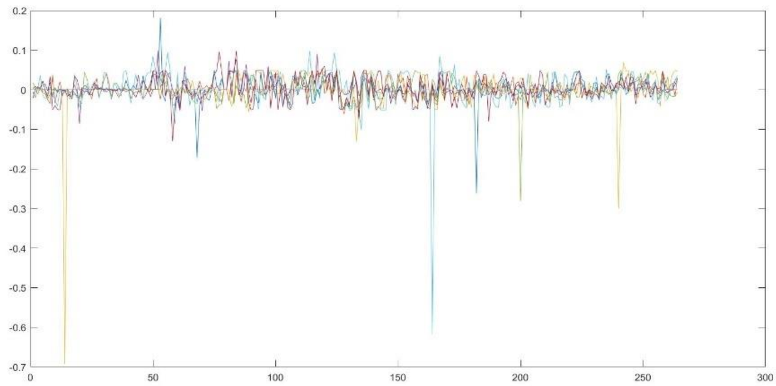

Fig. 1. Seven companies' stock returns.

The figure demonstrates the returns of the seven assets over time. The rhythm of returns, such as volatility or asymmetric tail behavior, characterized by negative skewness and Kurtosis, reflects underlying market dynamics. The number of returns jumps in the stock market affects each asset's efficiency. Therefore, this leads us to apply the VG process in assets' efficiency assessment, since its parameters could control skewness and Kurtosis.

The estimated parameters of the VG process based on the moment estimation method corresponding to each asset are recorded in *Table 2*. The VG factors are simulated by the Monte Carlo method, and then 1000 scenarios of assets' log-return are obtained.

Table 2. Estimated parameters of the VG process.

Asset	μ	θ	σ^2	ν
MSMI	-0.00086	0.003770	0.2650	0.003337
IKCO	0.003541	-0.002220	0.1325	0.060132
MAPN	-0.000910	0.005949	0.7685	0.017161
SMIF	0.007801	0.000696	0.1060	0.804985
APPE	-0.004330	-0.001190	0.1855	0.371983
ZOBI	0.003873	-0.001290	0.6890	0.170019
SEPP	0.007139	-0.001380	0.2385	3.692800

In *Table 3*, we present the results of VaR, CVaR, and mean return of simulated scenarios used by VG-based models, respectively. These data will be used as the input and the output of the models.

Table 3. VaR, CVaR, and Mean return of simulated scenarios of assets at confidence level $\beta = 0.90$.

Asset	Simulated data by VG parameters		
	VaR	CVaR	Mean-return
MSMI	0.2093	0.3983	-0.0001
IKCO	0.1842	0.3397	0.0022
MAPN	0.7885	1.5267	-0.0093
SMIF	0.1126	0.2318	0.0028
APPE	0.0873	0.1690	0.0017
ZOBI	0.6144	1.1879	-0.0026
SEPP	0.1898	0.3808	0.0032

Tables 4 and *5* show the inefficiency scores of the proposed models in the mean return-VaR and mean return-CVaR framework by *Models (6)* and *(7)*, respectively, under the *Directions (3)-(5)*.

Table 4. Inefficiency scores in the mean return-VaR and mean return-CVaR framework by Model (6) at $\beta = 0.90$.

Asset	Direction (3)		Direction (4)		Direction (5)	
	Mean-VaR	Mean -CVaR	Mean-VaR	Mean -CVaR	Mean-VaR	Mean -CVaR
MSMI	0.12	0.23	0.02	0.89	0.02	0.80
IKCO	0.08	0.14	0.01	0.62	0.01	0.56
MAPN	0.71	0.93	0.67	0.89	0.71	0.81
SMIF	0.00	0.00	0.00	0.00	0.00	0.00
APPE	0.00	0.00	0.00	0.00	0.00	0.00
ZOBI	0.53	0.85	0.36	0.74	0.40	0.68
SEPP	0.00	0.00	0.00	0.00	0.00	0.00

MAPN exhibits the highest inefficiency, with mean return-CVaR values of 0.93 (*Eq. (3)*), 0.89 (*Eq. (4)*), and 0.81 (*Eq. (5)*). Its extreme negative skewness (-8.8993) and high Kurtosis (110.8949) indicate a distribution prone to significant losses, corroborated by the simulated data's high VaR (0.7885), CVaR (1.5267), and negative mean return (-0.0093). The high Mean return-VaR (0.71 in D1 and D3, 0.67 in D2) further underscores MAPN's substantial risk exposure, contributing to its persistent inefficiency. ZOBI also shows significant inefficiency, with mean return-CVaR values of 0.85 (*Eq. (3)*), 0.74 (*Eq. (4)*), and 0.68 (*Eq. (5)*). Its negative skewness (-6.755) and high Kurtosis (83.3473) suggest a risk profile with frequent large negative returns, supported by simulated VaR (0.6144) and CVaR (1.1879) and a negative mean return (-0.0026). The mean return-VaR values (0.53 in (*Eq. (3)*), 0.36 in (*Eq. (4)*), 0.40 in (*Eq. (5)*) indicate varying risk exposure across scenarios.

MSMI demonstrates moderate inefficiency, with mean return-CVaR increasing sharply from 0.23 (*Eq. (3)*) to 0.89 (*Eq. (4)*), then slightly decreasing to 0.80 (*Eq. (5)*). Its negative skewness (-2.0873) and high Kurtosis (27.6953) suggest a risky distribution, with simulated VaR (0.2093) and CVaR (0.3983) indicating moderate risk. The near-zero mean return (-0.0001) and low mean return-VaR (0.12 in *Eq. (3)*), 0.02 in *Eq. (4),* and *Eq. (5)* suggest that inefficiency peaks in *Eq. (4)* due to heightened risk exposure. IKCO shows lower inefficiency, with mean return-CVaR values of 0.14 (*Eq. (3)*), 0.62 (*Eq. (4)*), and 0.56 (*Eq. (5)*). Its near-zero skewness (-0.0088) and moderate kurtosis (2.4636) indicate a more symmetric distribution, supported by a positive mean return (0.0022) and moderate simulated VaR (0.1842) and CVaR (0.3397). The low Mean-VaR (0.08 in *Eq. (3)*, 0.01 in *Eq. (4),* and *Eq. (5)*) suggests reduced risk in later scenarios, though inefficiency rises significantly in *Eq. (4)*.

SMIF, APPE, and SEPP report mean return-VaR and mean return-CVaR values of 0.00 across all directions, indicating no measurable inefficiency. SMIF's positive skewness (1.0106), moderate Kurtosis (9.4556), and positive mean return (0.0028) suggest a favorable risk profile, with low simulated VaR (0.1126) and CVaR (0.2318). APPE's negative skewness (-4.6107) and high Kurtosis (52.8753) indicate potential risk, yet its low simulated VaR (0.0873) and CVaR (0.1690) and positive mean return (0.0017) suggest effective risk management. SEPP's low skewness (-0.2091), Kurtosis (4.0696), positive mean return (0.0032), and moderate VaR (0.1898) and CVaR (0.3808) support its PDDF efficiency.

The table shows a general trend of increasing mean return-CVaR from *Eq. (3)* to *Eq. (4)* for MSMI (0.23 to 0.89) and IKCO (0.14 to 0.62), aligning with the expected trend of increasing inefficiency. However, MAPN and ZOBI exhibit a slight decrease in mean return-CVaR from *Eq. (3)* to *Eq. (4)* (MAPN: 0.93 to 0.81, ZOBI: 0.85 to 0.68), which contradicts the expected trend and may indicate data inconsistencies, as previously noted, potentially due to their high negative skewness and Kurtosis distorting risk metrics. The slight decrease in *Eq. (3)* for MSMI (0.89 to 0.80) and IKCO (0.62 to 0.56) suggests a partial alignment with the expected trend, with peak inefficiency in *Eq. (4)*. mean return-VaR values generally decrease from *Eq. (3)* to *Eq. (4)* for MSMI (0.12 to 0.02), IKCO (0.08 to 0.01), and ZOBI (0.53 to 0.36/0.40), indicating reduced risk at a single confidence level. In contrast, MAPN's mean return-VaR remains high (0.71 to 0.67/0.71). The static zero values for SMIF, APPE, and SEPP across directions reinforce their consistent efficiency.

The increasing mean return-CVaR from *Eq. (3)* to *Eq. (4)* for MSMI and IKCO supports the expected trend of rising inefficiency. Still, the slight decrease in *Eq. (5)* for most assets suggests potential data inconsistencies or scenario-specific factors. Decision makers or portfolio managers should validate the data and consider dynamic strategies to address varying inefficiencies across scenarios.

Overall, the findings indicate that assessing asset efficiency under different frameworks incorporating risk-based measures such as VaR and CVaR can lead to varying outcomes. In particular, directions that focus primarily on maximum values of conditional risk and return tend to impose stricter evaluations, *Direction (3)*. This highlights the fact that the choice of evaluation criteria, potential data inconsistencies, and analytical framework or scenario-specific can significantly influence the final judgment regarding an asset's efficiency. Factors such as Skewness and Kurtosis can also affect the evaluation of asset efficiency.

Therefore, to achieve a comprehensive and realistic assessment of asset performance, it is essential to consider the differences among evaluation directions and risk measures. Also, Decision makers or portfolio managers should validate the data and consider dynamic strategies to address varying inefficiencies across scenarios.

The inefficiency scores obtained by *Model (7)* are shown in *Table 5*.

Table 5. Inefficiency scores in the mean return-CVaR framework by Model (P2) at $\beta = 0.90$.

Asset	Direction (3) Mean-CVaR	Direction (4) Mean-CVaR	Direction (5) Mean-CVaR
MSMI	0.25	0.91	0.83
IKCO	0.18	0.69	0.61
MAPN	0.94	0.90	0.86
SMIF	0.00	0.00	0.00
APPE	0.00	0.00	0.00
ZOBI	0.88	0.78	0.70
SEPP	0.00	0.00	0.00

MAPN exhibits the highest inefficiency, with a mean return-CVaR of 0.94 across all directions. This aligns with its extreme negative skewness (-8.8993) and high Kurtosis (110.8949), indicating a distribution with significant downside risk and heavy tails, which likely contributes to its persistent inefficiency. The high CVaR (1.5267) from the simulated data further confirms MAPN's elevated risk profile and negative mean return (-0.0093), underscoring its inefficiency. ZOBI also shows significant inefficiency, with a mean return-CVaR of 0.88 across all directions. Its negative skewness (-6.755) and high Kurtosis (83.3473) suggest a similar risk profile to MAPN, with substantial downside risk and fat-tailed returns. The simulated CVaR (1.1879) reinforces ZOBI's high-risk characteristics, contributing to its inefficiency.

MSMI and IKCO display moderate inefficiency, with mean return-CVaR values of 0.25 and 0.18, respectively, across all directions. MSMI's significant negative skewness (-2.0873) and high Kurtosis (27.6953) suggest a riskier distribution, which aligns with its higher inefficiency compared to IKCO, which has near-zero skewness (-0.0088) and moderate Kurtosis (2.4636). The simulated data show MSMI's CVaR (0.3983) slightly higher than IKCO's (0.3397), consistent with their relative inefficiency levels.

SMIF, APPE, and SEPP report mean return-CVaR values of 0.00 across all directions, indicating no measurable inefficiency, and they are PDDF-efficient. SMIF's positive skewness (1.0106) and moderate Kurtosis (9.4556) suggest a more favorable risk profile, while APPE's negative skewness (-4.6107) and high Kurtosis (52.8753) indicate potential risk that is not reflected in the inefficiency score. SEPP's low skewness (-0.2091) and Kurtosis (4.0696) suggest a stable distribution.

Unlike the expected trend of increasing inefficiency from *Eq. 3* to *Eq. 5* noted in prior discussions, the mean return-CVaR values in this table remain constant across all directions. This static behavior suggests that the inefficiency of each asset is unaffected by changes in the conditions represented by *Eqs. (3)-(5).* It may indicate that the scenarios are similar in their impact on inefficiency or that the mean return-CVaR metric, as calculated here, does not capture directional variations. The consistency of high inefficiency for MAPN and ZOBI, combined with their extreme skewness and Kurtosis, suggests that their inefficiency is driven by inherent distributional characteristics rather than scenario-specific factors. Similarly, the zero inefficiency for SMIF, APPE, and SEPP may reflect a structural feature of these assets, such as hedging or low exposure to risk factors in all directions.

The table reveals significant variation in asset inefficiency, with MAPN and ZOBI exhibiting the highest levels (mean return-CVaR of 0.94 and 0.88), driven by their extreme negative skewness, high Kurtosis, and elevated VaR/CVaR values. MSMI and IKCO show moderate inefficiency, while SMIF, APPE, and SEPP report zero inefficiency, potentially reflecting efficiency or data limitations. The static mean return-CVaR values across *Eqs. (3)-(5)* suggest that inefficiency is consistent across scenarios, which contrasts with the expected trend of increasing inefficiency. This discrepancy, combined with the distributional characteristics from the skewness and kurtosis table, underscores the need for further data validation and careful portfolio management to address the high inefficiency of assets like MAPN and ZOBI while leveraging the apparent stability of SMIF, APPE, and SEPP.

4 | Conclusion

Portfolio diversification remains essential for optimizing investment outcomes across varied assets. This study employs DEA with DDFs and the VG process to assess asset efficiency, enabling investors to optimize portfolios by modeling leptokurtic, skewed, and fat-tailed return distributions. The first model reveals that asset efficiency varies with the mean return-risk framework, with stricter evaluations under high-risk and return criteria, influenced by skewness and Kurtosis. The second model's static mean return-CVaR values across directions suggest inefficiencies driven by intrinsic distributional properties rather than scenario-specific factors.

In conclusion, the observed patterns in asset returns mention the complex interplay of risk and return, driven by market-specific and external factors. These findings emphasize the need for robust analytical models, such as those incorporating the VG process, to capture non-normal return distributions and inform efficient portfolio construction to help the investors. Further investigation into the drivers of these patterns is essential for optimizing investment decisions in volatile market environments. Future research aims to integrate DEA with machine learning techniques to enhance predictive capabilities and improve decision-making.

References

[1] Markowitz, H. (1952). Portfolio selection. *The journal of finance*, *7*(1), 77–91. https://doi.org/10.1111/j.1540-6261.1952.tb01525.x

[2] Markowitz, H. M. (2008). *Portfolio selection: Efficient diversification of investments*. Yale University Press. https://books.google.com/books?id=GZDyAAAAQBAJ

[3] Charnes, A., Cooper, W. W., & Rhodes, E. (1978). Measuring the efficiency of decision making units. *European journal of operational research*, *2*(6), 429–444. https://doi.org/10.1016/0377-2217(78)90138-8

[4] Morey, M. R., & Morey, R. C. (1999). Mutual fund performance appraisals: A multi-horizon perspective with endogenous benchmarking. *Omega*, *27*(2), 241–258. https://doi.org/10.1016/S0305-0483(98)00043-7

[5] Liu, W., Zhou, Z., Liu, D., & Xiao, H. (2015). Estimation of portfolio efficiency via DEA. *Omega*, *52*, 107–118. https://doi.org/10.1016/j.omega.2014.11.006

[6] Briec, W., & Kerstens, K. (2009). Multi-horizon Markowitz portfolio performance appraisals: A general approach. *Omega*, *37*(1), 50–62. https://doi.org/10.1016/j.omega.2006.07.007

[7] Joro, T., & Na, P. (2006). Portfolio performance evaluation in a mean–variance–skewness framework. *European journal of operational research*, *175*(1), 446–461. https://doi.org/10.1016/j.ejor.2005.05.006

[8] Kerstens, K., Mounir, A., & Van de Woestyne, I. (2011). Geometric representation of the mean–variance–skewness portfolio frontier based upon the shortage function. *European journal of operational research*, *210*(1), 81–94. https://doi.org/10.1016/j.ejor.2010.09.014

[9] Chambers, R. G., Chung, Y., & Färe, R. (1998). Profit, directional distance functions, and nerlovian efficiency. *Journal of optimization theory and applications*, *98*(2), 351–364. https://doi.org/10.1023/A:1022637501082

[10] Fukuyama, H., & Weber, W. L. (2004). Economic inefficiency measurement of input spending when decision-making units face different input prices. *Journal of the operational research society*, *55*(10), 1102–1110. https://doi.org/10.1057/palgrave.jors.2601750

[11] Shephard, R. W. (1953). *Cost and production functions*. Princeton University Press. https://books.google.com/books?id=I7u6BgNZuacC

[12] Shephard, R. W. (1970). Theory of cost and production functions. *Princeton University Press*. https://B2n.ir/nn8667

[13] Sahoo, B. K., Mehdiloozad, M., & Tone, K. (2014). Cost, revenue and profit efficiency measurement in DEA: A directional distance function approach. *European journal of operational research*, *237*(3), 921–931. https://doi.org/10.1016/j.ejor.2014.02.017

[14] Tavana, M., Izadikhah, M., Toloo, M., & Roostaee, R. (2021). A new non-radial directional distance model for data envelopment analysis problems with negative and flexible measures. *Omega*, *102*, 102355. https://doi.org/10.1016/j.omega.2020.102355

[15] Kraus, A., & Litzenberger, R. H. (1976). Skewness preference and the valuation of risk assets. *The journal of finance*, *31*(4), 1085–1100. https://doi.org/10.2307/2326275

[16] Artzner, P., Delbaen, F., Eber, J. M., & Heath, D. (1997). Thinking coherently risk. *Journal of risk*, *10*(68), 71. https://people.math.ethz.ch/~delbaen/ftp/preprints/CoherentMF.pdf

[17] Artzner, P., Delbaen, F., Eber, J. M., & Heath, D. (1999). Coherent measures of risk. *Mathematical finance*, *9*(3), 203–228. https://doi.org/10.1111/1467-9965.00068

[18] Andersson, F., Mausser, H., Rosen, D., & Uryasev, S. (2001). Credit risk optimization with conditional value-at-risk criterion. *Mathematical programming*, *89*(2), 273–291. https://doi.org/10.1007/PL00011399

[19] Alexander, S., Coleman, T. F., & Li, Y. (2003). Derivative portfolio hedging based on CVaR. *New risk measures in investment and regulation: Wiley*. https://cs.uwaterloo.ca/~yuying/papers/cvarBook.pdf

[20] Yu, J., Yang, X., & Li, S. (2009). Portfolio optimization with CVaR under VG process. *Research in international business and finance*, *23*(1), 107–116. https://doi.org/10.1016/j.ribaf.2008.07.004

[21] Madan, D. B., Carr, P. P., & Chang, E. C. (1998). The variance gamma process and option pricing. *Review of finance*, *2*(1), 79–105. https://doi.org/10.1023/A:1009703431535

[22] Bergmann, D. R., Ferreira Savoia, J. R., Felisoni de Angelo, C., Contani, E. A. do R., & Da Silva, F. L. (2018). Portfolio management with tail dependence. *Applied economics*, *50*(51), 5510–5520. https://doi.org/10.1080/00036846.2018.1487000

[23] Xu, C., Yang, F., & Sun, Y. (2019). An efficient acceleration monte carlo simulation for pricing Asian option under variance gamma process by splitting. *International journal of computer mathematics*, *96*(12), 2522–2532. https://doi.org/10.1080/00207160.2019.1577399

[24] Mirsadeghpour Zoghi, S. M., Sanei, M., Tohidi, G., Banihashemi, S., & Modarresi, N. (2023). Performance evaluation of investment with DEA models under pure jump processes. *Concurrency and computation: practice and experience*, *35*(3), e7502. https://doi.org/10.1002/cpe.7502

[25] Rockafellar, R. T., & Uryasev, S. (2000). Optimization of conditional value-at-risk. *The journal of risk*, *2*(3), 21–41. https://doi.org/10.21314/jor.2000.038

[26] Lynn, M. (2014). *Are you brave enough to invest in Iran?* https://B2n.ir/qt2434