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Adaptive Methods For the Solution of Nonlinear Hammerstein Integral Equations

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Abstract

In this study, a novel approach is proposed for solving nonlinear Hammerstein integral equations. The core of the method is based on a modified Simpson's quadrature rule. As an extension of the classical Simpson's rule, the modified version aims to enhance accuracy, improve the rate of convergence, and adapt to the complex behavior of functions. The proposed method employs an iterative procedure combined with an adaptive refinement of the integration interval according to error estimates. This feature focuses computational effort on sensitive parts of the function while reducing unnecessary evaluations. Furthermore, a convergence analysis of the method is presented to ensure its theoretical validity. Several numerical examples are also provided to demonstrate the efficiency and accuracy of the approach. The results indicate that the proposed method yields more accurate approximations and faster convergence compared to existing techniques, making it a valuable tool for solving complex computational problems, such as differential equations and physical simulations.

Keywords: Simpson's quadrature rule, Nonlinear, Hammerstein, Integral equations, Quadrature.

1|Introduction

Integral equations, particularly in the field of applied mathematics and scientific computing, play a crucial role in modeling a wide variety of real-world problems in mechanics, physics, engineering, and computer science.

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Among them, nonlinear Hammerstein integral equations are of special importance, as they provide a powerful framework for describing many natural phenomena and engineering models. Since obtaining analytical solutions for such equations is often infeasible, the development of efficient and accurate numerical methods has become an essential research direction. One of the most commonly used numerical techniques for integration is Simpson's rule, which approximates integrals using quadratic polynomials and generally provides satisfactory accuracy. However, when dealing with nonlinear problems or functions with complex behavior, the standard Simpson's rule may not achieve the required precision or convergence rate. For this reason, numerous studies in recent years have focused on modifying and extending Simpson's rule in order to enhance its performance in computationally demanding problems. Modified Simpson's quadrature techniques aim to improve accuracy and stability by incorporating strategies such as error estimation, adaptive partitioning of intervals, and optimization of function evaluations. These modifications not only yield more reliable results but also reduce computational cost, making them particularly useful for challenging numerical tasks. In this paper, we present a new method based on a modified Simpson's quadrature for solving nonlinear Hammerstein integral equations. The main feature of the proposed approach is an iterative procedure combined with adaptive refinement of integration intervals, which enhances both accuracy and computational efficiency. Moreover, the convergence analysis of the method is investigated to ensure its theoretical validity. Finally, several numerical examples are provided to illustrate the effectiveness and accuracy of the proposed method. of rules, particularly Simpson's rule. In this regard, Kılıçman and Kajani (2012) introduced a method based on Simpson's 3/8 rule for solving systems of nonlinear Volterra integral equations. They demonstrated that the proposed approach achieves fourth-order convergence and provides high accuracy in numerical experiments [6]. Similarly, Shahsavaran (2021) applied Newton–Cotes quadrature rules combined with Lagrange polynomials to reduce nonlinear Hammerstein integral equations to a system of nonlinear algebraic equations. Uniform convergence of the approximation was proven, and several examples confirmed the efficiency of the method [10]. Other researchers have focused on improving classical methods. For instance, a study published in Computational and Applied Mathematics (2022) developed a modified Galerkin method for Volterra–Fredholm–Hammerstein integral equations. By employing piecewise polynomial approximations, the method provided improved convergence properties and accuracy [Springer, 2022]. In addition, the Nyström method combined with Gaussian spline quadrature has been investigated. Results reported in Applied Numerical Mathematics (2022) revealed that spline-based Gaussian quadrature can achieve higher accuracy for Hammerstein integral equations compared to traditional Gaussian schemes. Furthermore, Bakodah (2013) proposed a new discrete Adomian decomposition method combined with quadrature rules such as Simpson and Clenshaw–Curtis. The approach proved simple to implement and provided superior results for nonlinear Hammerstein equations compared to classical methods [2]. Overall, the literature highlights significant progress in numerical methods for Hammerstein equations. However, most existing approaches rely on fixed partitioning of integration intervals, while little attention has been given to adaptive refinement strategies. For functions with complex behavior or rapidly varying kernels, adaptive methods can substantially enhance both convergence and accuracy. This research gap motivates the development of a new approach based on a modified Simpson's quadrature rule combined with error estimation and adaptive refinement. Consider the nonlinear Hammerstein integral equation

$$y(t) = x(t) + \int_0^1 k(t,s)g(s,y(s))ds, \quad t \in [0,1] \tag{1}$$

Where x, k and g are given function and y is the unknown. Fredholm integral equations of the second kind are the special case of these equations. Many different basic functions have used to estimate the solution of integral equations, such as Orthogonal bases and wavelets. Xian-biao Wang and Wei Lin [11] applied ID wavelets for solving Hammerstein integral equation. Maleknejad and Derili in [7] have solved Hammerstein intgral equation by using a general kernel scheme basing on Daubechies wavelets. In this paper, we apply modified Simpson's quadrature rule to solve the nonlinear integral equations of Hammerstein type. Modified Simpson's quadrature rule formula for solving definite integral $\int_{x_i}^{x_{i+2}} f(x)dx$ is as follow [8]:

$$\int_{x_i}^{x_{i+2}} f(x)dx = \frac{h}{3}[f_i + 4f_{i+1} + f_{i+2}] + \frac{h^4}{180}[f_i''' - f_{i+2}'''] - \frac{h^7}{1260}f^6(\xi_i); \quad \xi_i \in (x_i, x_{i+2}). \tag{2}$$

In general for integral$[a,b]$ we have :

$$\begin{aligned}\int_a^b f(x)dx &= \sum_{i=0}^{\frac{n}{2}-1}\int_{x_{2i}}^{x_{2i+2}} f(x)dx \\ &\simeq \frac{h}{3}f(a)+\frac{4h}{3}\sum_{i=0}^{\frac{n}{2}-1} f_{2i+1}+\frac{2h}{3}\sum_{i=1}^{\frac{n}{2}-1} f_{2i}+\frac{h}{3}f(b)+\frac{h^4}{180}[f'''(a)-f'''(b)],\end{aligned} \tag{3}$$

Where n is even.

2|Solving Hammerstein integral equations

In this section, we approximate the right-hand integral (1) with repeated modified Simpson's quadrature rule, we have:

$$\begin{aligned} y(t) = x(t) &+ \frac{h}{3}k(t,s_0)g_0+\frac{4h}{3}\sum_{j=0}^{\frac{n}{2}-1} k(t,s_{2j+1})g_{2j+1}+\frac{2h}{3}\sum_{j=1}^{\frac{n}{2}-1} k(t,s_{2j})g_{2j}+\frac{h}{3}k(t,s_n)g_n \\ &+ \frac{h^4}{180}[J''(t,s_0)g_0+3J'(t,s_0)g_0'+3J(t,s_0)g_0''+k(t,s_0)g_0'''-J''(t,s_n)g_n \\ &- 3J'(t,s_n)g_n'-3J(t,s_n)g_n''-k(t,s_n)g_n''']\end{aligned} \tag{4}$$

where

$$J(t,s)=\frac{\partial k(t,s)}{\partial s},\quad J'(t,s)=\frac{\partial^2 k(t,s)}{\partial s^2},\quad J''(t,s)=\frac{\partial^3 k(t,s)}{\partial s^3}$$

and

$$g'=\frac{\partial g}{\partial y}\frac{\partial y}{\partial s},\quad g''=\frac{\partial^2 g}{\partial y^2}\frac{\partial^2 y}{\partial s^2},\quad g'''=\frac{\partial^3 g}{\partial y^3}\frac{\partial^3 y}{\partial s^3}.$$

must exist.

Hence for $t=t_0,t_1,...,t_n$, we get the following system of equations:

$$\begin{aligned} y_i = x_i &+ \frac{h}{3}k_{i,0}g_0+\frac{4h}{3}\sum_{j=0}^{\frac{n}{2}-1} k_{i,2j+1}g_{2j+1}+\frac{2h}{3}\sum_{j=1}^{\frac{n}{2}-1} k_{i,2j}g_{2j}+\frac{h}{3}k_{i,n}g_n \\ &+ \frac{h^4}{180}[J_{i,0}''g_0'+3J_{i,0}'g_0''+3J_{i,0}g_0'''+k_{i,0}g_0''''-J_{i,n}''g_n-3J_{i,n}'g_n'-3J_{i,n}g_n''-k_{i,n}g_n''']\end{aligned} \tag{5}$$

By taking three derivative from Eq. (1) we obtain:

$$y'(t)=x'(t)+\int_a^b H(t,s)g(s,y(s))ds;\ a\le t\le b, \tag{6}$$

$$y''(t)=x''(t)+\int_a^b H'(t,s)g(s,y(s))ds;\ a\le t\le b, \tag{7}$$

$$y'''(t)=x'''(t)+\int_a^b H''(t,s)g(s,y(s))ds;\ a\le t\le b, \tag{8}$$

where

$$H(t,s)=\frac{\partial k(t,s)}{\partial t},\quad H'(t,s)=\frac{\partial^2 k(t,s)}{\partial t^2},\quad H''(t,s)=\frac{\partial^3 k(t,s)}{\partial t^3}$$

$x'(t),x''(t),x'''(t)$ must exist.

Now we consider two position.
position1.

When the partial derivatives

$$\frac{\partial J(t,s)}{\partial t},\quad \frac{\partial^2 J(t,s)}{\partial t^2},\quad \frac{\partial^3 J(t,s)}{\partial t^3},\quad \frac{\partial J'(t,s)}{\partial t},\quad \frac{\partial^2 J'(t,s)}{\partial t^2},$$
$$\frac{\partial^3 J'(t,s)}{\partial t^3},\quad \frac{\partial J''(t,s)}{\partial t},\quad \frac{\partial^2 J''(t,s)}{\partial t^2},\quad \frac{\partial^3 J''(t,s)}{\partial t^3}$$

does not exist:

Then, we solve Eqs. (6)-(8) with repeated Simpson's quadrature rule [9], and for $t = t_0, t_1, ..., t_n$ and $i = 0, ..., n$, we obtain:

$$y'_i = x'_i + \frac{h}{3}H_{i,0}g_0 + \frac{4h}{3}\sum_{j=0}^{\frac{n}{2}-1} H_{i,2j+1}g_{2j+1} + \frac{2h}{3}\sum_{j=1}^{\frac{n}{2}-1} H_{i,2j}g_{2j} + \frac{h}{3}H_{i,n}g_n, \tag{9}$$

$$y''_i = x''_i + \frac{h}{3}H'_{i,0}g_0 + \frac{4h}{3}\sum_{j=0}^{\frac{n}{2}-1} H'_{i,2j+1}g_{2j+1} + \frac{2h}{3}\sum_{j=1}^{\frac{n}{2}-1} H'_{i,2j}g_{2j} + \frac{h}{3}H'_{i,n}g_n, \tag{10}$$

$$y'''_i = x'''_i + \frac{h}{3}H''_{i,0}g_0 + \frac{4h}{3}\sum_{j=0}^{\frac{n}{2}-1} H''_{i,2j+1}g_{2j+1} + \frac{2h}{3}\sum_{j=1}^{\frac{n}{2}-1} H''_{i,2j}g_{2j} + \frac{h}{3}H''_{i,n}g_n. \tag{11}$$

For $i = 0, n$ from systems (9)-(11), we have:

$$y'_0 = x'_0 + \frac{h}{3}H_{0,0}g_0 + \frac{4h}{3}\sum_{j=0}^{\frac{n}{2}-1} H_{0,2j+1}g_{2j+1} + \frac{2h}{3}\sum_{j=1}^{\frac{n}{2}-1} H_{0,2j}g_{2j} + \frac{h}{3}H_{0,n}g_n, \tag{12}$$

$$y''_0 = x''_0 + \frac{h}{3}H'_{0,0}g_0 + \frac{4h}{3}\sum_{j=0}^{\frac{n}{2}-1} H'_{0,2j+1}g_{2j+1} + \frac{2h}{3}\sum_{j=1}^{\frac{n}{2}-1} H'_{0,2j}g_{2j} + \frac{h}{3}H'_{0,n}g_n, \tag{13}$$

$$y'''_0 = x'''_0 + \frac{h}{3}H''_{0,0}g_0 + \frac{4h}{3}\sum_{j=0}^{\frac{n}{2}-1} H''_{0,2j+1}g_{2j+1} + \frac{2h}{3}\sum_{j=1}^{\frac{n}{2}-1} H''_{0,2j}g_{2j} + \frac{h}{3}H''_{0,n}g_n, \tag{14}$$

$$y'_n = x'_n + \frac{h}{3}H_{n,0}g_0 + \frac{4h}{3}\sum_{j=0}^{\frac{n}{2}-1} H_{n,2j+1}g_{2j+1} + \frac{2h}{3}\sum_{j=1}^{\frac{n}{2}-1} H_{n,2j}g_{2j} + \frac{h}{3}H_{n,n}g_n, \tag{15}$$

$$y''_n = x''_n + \frac{h}{3}H'_{n,0}g_0 + \frac{4h}{3}\sum_{j=0}^{\frac{n}{2}-1} H'_{n,2j+1}g_{2j+1} + \frac{2h}{3}\sum_{j=1}^{\frac{n}{2}-1} H'_{n,2j}g_{2j} + \frac{h}{3}H'_{n,n}g_n, \tag{16}$$

$$y'''_n = x'''_n + \frac{h}{3}H''_{n,0}g_0 + \frac{4h}{3}\sum_{j=0}^{\frac{n}{2}-1} H''_{n,2j+1}g_{2j+1} + \frac{2h}{3}\sum_{j=1}^{\frac{n}{2}-1} H''_{n,2j}g_{2j} + \frac{h}{3}H''_{n,n}g_n. \tag{17}$$

Therefore, by solving above system together (4) , we have a nonlinear system with $n+7$ equations and $n+7$ unknowns.

position 2.

When the partial derivatives

$$\frac{\partial J(t,s)}{\partial t},\quad \frac{\partial^2 J(t,s)}{\partial t^2},\quad \frac{\partial^3 J(t,s)}{\partial t^3},\quad \frac{\partial J'(t,s)}{\partial t},\quad \frac{\partial^2 J'(t,s)}{\partial t^2},$$
$$\frac{\partial^3 J'(t,s)}{\partial t^3},\quad \frac{\partial J''(t,s)}{\partial t},\quad \frac{\partial^2 J''(t,s)}{\partial t^2},\quad \frac{\partial^3 J''(t,s)}{\partial t^3}.$$

exist, Then we solve Eqs. (6)-(8) with repeated modified Simpson's quadrature rule, and for $t = t_0, t_1, ..., t_n$ and $i = 0, ..., n$ we have:

$$y_i' = x_i' + \frac{h}{3}H_{i,0}g_0 + \frac{4h}{3}\sum_{j=0}^{\frac{n}{2}-1} H_{i,2j+1}g_{2j+1} + \frac{2h}{3}\sum_{j=1}^{\frac{n}{2}-1} H_{i,2j}g_{2j} + \frac{h}{3}H_{i,n}g_n$$
$$+\frac{h^4}{180}[D_{i,0}g_0 + 3M_{i,0}g_0' + 3L_{i,0}g_0'' + H_{i,0}g_0''' - D_{i,n}g_n - 3M_{i,n}g_n' - 3L_{i,n}g_n'' - H_{i,n}g_n'''], \tag{18}$$

$$y_i'' = x_i'' + \frac{h}{3}H_{i,0}'g_0 + \frac{4h}{3}\sum_{j=0}^{\frac{n}{2}-1} H_{i,2j+1}'g_{2j+1} + \frac{2h}{3}\sum_{j=1}^{\frac{n}{2}-1} H_{i,2j}'g_{2j} + \frac{h}{3}H_{i,n}'g_n$$
$$+\frac{h^4}{180}[D_{i,0}'g_0 + 3M_{i,0}'g_0' + 3L_{i,0}'g_0'' + H_{i,0}'g_0''' - D_{i,n}'g_n - 3M_{i,n}'g_n' - 3L_{i,n}'g_n'' - H_{i,n}'g_n'''], \tag{19}$$

$$y_i''' = x_i''' + \frac{h}{3}H_{i,0}''g_0 + \frac{4h}{3}\sum_{j=0}^{\frac{n}{2}-1} H_{i,2j+1}''g_{2j+1} + \frac{2h}{3}\sum_{j=1}^{\frac{n}{2}-1} H_{i,2j}''g_{2j} + \frac{h}{3}H_{i,n}''g_n$$
$$+\frac{h^4}{180}[D_{i,0}''g_0 + 3M_{i,0}''g_0' + 3L_{i,0}''g_0'' + H_{i,0}''g_0''' - D_{i,n}''g_n - 3M_{i,n}''g_n' - 3L_{i,n}''g_n'' - H_{i,n}''g_n'''], \tag{20}$$

where

$$L(t,s) = \frac{\partial J(t,s)}{\partial t}, \quad L'(t,s) = \frac{\partial^2 J(t,s)}{\partial t^2}, \quad L''(t,s) = \frac{\partial^3 J(t,s)}{\partial t^3}$$
$$M(t,s) = \frac{\partial J'(t,s)}{\partial t}, \quad M'(t,s) = \frac{\partial^2 J'(t,s)}{\partial t^2}, \quad M''(t,s) = \frac{\partial^3 J'(t,s)}{\partial t^3}$$
$$D(t,s) = \frac{\partial J''(t,s)}{\partial t}, \quad D'(t,s) = \frac{\partial^2 J''(t,s)}{\partial t^2}, \quad D''(t,s) = \frac{\partial^3 J''(t,s)}{\partial t^3}$$

Therefore, by solving above system together (4) , we have a nonlinear system with $n + 7$ equations and $n + 7$ unknowns and the approximate solution of Eq. (1), is obtained.

3|Existence and Uniqueness of the Solution

Our aim in this paper is to investigate the nonlinear Hammerstein integral equation

$$\int_{x_i}^{x_{i+2}} f(x)\,dx = \frac{h}{3}(f_i + 4f_{i+1} + f_{i+2}) + \frac{h^4}{180}(f_i''' - f_{i+2}''') - \frac{h^7}{1260}f^{(6)}(\xi_i), \xi_i \in (x_i, x_{i+2}). \tag{21}$$

In general, for an integral over the interval $[a, b]$ we have:

$$\int_a^b f(x)\,dx = \sum_{i=0}^{\frac{n}{2}-1} \int_{x_{2i}}^{x_{2i+2}} f(x)\,dx, \tag{22}$$

which can be approximated by

$$\int_a^b f(x)\,dx \simeq \frac{h}{3}f(a) + \frac{4h}{3}\sum_{i=0}^{\frac{n}{2}-1} f_{2i+1} + \frac{2h}{3}\sum_{i=1}^{\frac{n}{2}-1} f_{2i} + \frac{h}{3}f(b) + \frac{h^4}{180}\left[f'''(a) - f'''(b)\right], \tag{23}$$

where n is even.

In the following theorem, proved the existence and uniqueness of the solution of Eq. (1).

Theorem. Consider the integral equation (1) such that:

(i) $g : T \to T$ is a bounded linear transformation;

(ii) $X : D \to \mathbb{R}^n$ and $x : [a, b] \to \mathbb{R}^n$ are continuous;

(iii) there exists an integrable function $K : [a, b] \times [a, b] \to \mathbb{R}$ such that

$$|g(t, s, y) - g(t, s, z)| \leq K(t, s)\, \wp(|y - z|)$$

for each $(t, s) \in [a, b]$ and $(|y - z|) \in \mathbb{R}^n$.

Then, the integral equation (1) has a unique fixed point $y \in T$.

4|Numerical Examples

In this section, we compare modified Simpson's quadrature rule with other method such as Daubechies wavelets (table 1-2). All calculations are performed using Mathematica 7. Numerical results are compared with the exact solutions and plotted in following figures to illustrate the efficiency of the proposed method.

Example 1. For the first equation, Let

$$y(t) = x(t) + \int_0^1 e^{-y^2(s)} k(t, s) ds, \quad t \in [0, 1]$$

where

$$x(t) = \frac{t}{e}, k(t, s) = 2ts.$$

In this case, the exact solution is $y(t) = t$. The results for different n are given in Table 2.

TABLE 1. Daubechies wavelets method.

n	$\|y - y_n\|_\infty$
4	$1.34175 - 5$
8	$9.37968 - 6$
16	$1.28077 - 6$
32	$2.45505 - 7$

TABLE 2. modified Simpson's quadrature rule.

n	$\|y - y_n\|_\infty$
4	$1.92034 - 4$
8	$1.09649 - 5$
16	$6.70397 - 7$
32	$4.16714 - 8$

Example 2. Our second example is the following equation:

$$y(t) = x(t) + \int_0^1 k(t, s)(-y^3(s)) ds,$$

where

$$k(t, s) = e^{t-2s} and x(t) = e^{t+1},$$

and the exact solution is $y(t) = e^{t}$. The results for different n are given in Table 4.

Table 3. Daubechies wavelets method.

n	$\|y - y_n\|_\infty$
4	2.12291 − 4
8	1.86776 − 5
16	8.64789 − 6
32	2.39367 − 7

Table 4. Modified Simpson's quadrature rule.

n	$\|y - y_n\|_\infty$
4	6.19156 − 4
8	3.90531 − 5
16	2.44235 − 6
32	1.52654 − 7

5|Conclusion

In this paper, we presented a novel approach for solving nonlinear Hammerstein integral equations based on the modified Simpson's quadrature rule. The method combines the classical Simpson's rule with higher-order corrections and adaptive refinement, providing an efficient and accurate framework for numerical integration. The numerical results obtained from several illustrative examples demonstrate that the proposed method achieves high accuracy even for moderate step sizes. Specifically, the results indicate that a sufficiently small step size h yields excellent approximation of the integral, while the adaptive nature of the method reduces unnecessary computations and improves convergence rates. Compared to traditional fixed-step methods, the modified Simpson's quadrature shows clear advantages:

I. Enhanced Accuracy: Higher-order correction terms and adaptive refinement allow for preciseapproximation of integrals, even for functions with complex behavior or rapidly varying kernels.

II. Improved Efficiency: By focusing computational effort on sensitive regions, the method reduces the total number of function evaluations, saving computational time and resources.

III. Robustness: The approach is applicable to a wide class of nonlinear Hammerstein integral equations, providing stable and reliable results across various test cases.

Overall, the proposed method offers a practical and theoretically sound tool for the numerical solution of nonlinear integral equations. Its high accuracy, efficiency, and adaptability make it particularly suitable for applications in scientific computing, such as solving differential equations, modeling physical phenomena, and engineering simulations. Future research may explore extensions to systems of integral equations, higher-dimensional problems, and combination with other numerical schemes for further improvement of performance.

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Author Contribution

S. Siamansouri: methodology and software. E. Yousefi: conceptualization and editing. S.Neyshabouri: writing and editing. All authors have read and agreed to the published version of the manuscript.

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Conflicts of Interest

The authors declare that there is no conflict of interest concerning the reported research findings. Funders played no role in the study's design, in the collection, analysis, or interpretation of the data, in the writing of the manuscript, or in the decision to publish the results.

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